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Optimal Controller Placement in Modal Control of Complex Systems*

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Within the framework of modal control of large systems, a simple approach is advanced for the determination of optimal control configuration under an energy constraint, i.e., optimal locations of a limited number of controllers such that the total energy requirement for control is minimized. It is shown that the resulting design criterion is a simple function of projections of the control matrix onto components of eigenvectors associated with the affected eigenvalues. Furthermore, it is applicable to both single-input and multi-input systems. Systems possessing distinct complex eigenvalues are considered but the approach is equally applicable to other types of systems. Examples show that the minimum-energy control configuration also tends to be the most effective in terms of accomplishing control objectives.

I. INTRODUCTION

Active feedback control of large systems such as space structures [1, 2], tall buldings [3, 4], and chemical processes [5] introduces a number of difficult problems in the applications of modern control theory. Because of their large dimensions, standard control design offers only limited help in implementation of control to these systems. One of the important problems is that of optimal control configuration, that is, the determination of appropriate locations of controllers when, due to practical and economic considerations, only a limited number of them are available.

Consider the standard state-space systems equation

$$\dot{x} = Ax + Bu \tag{1}$$

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where x is the state vector, A is the system matrix, B is the control matrix and u is the control vector. The aforementioned problem is then one of determining the optimal form of B given a restricted number of controllers, optimal in the sense that some performance index is optimized. This problem is considered in this paper within the framework of modal control theory. Specifically, given the control objective, an optimal B is to be found among all feasible choices. The performance index considered here is an energy criterion which takes the form

$$E = \int_0^{T_0} u^T Q u \, dt \tag{2}$$

where Q is a positive definite weighting matrix.

In order to proceed, a general design procedure is needed for modal control of systems having complex eigenvalues. In the next section, a sequential design procedure is first developed. It is followed by the determination of criteria for optimal location of controllers. Results will be presented for the case

$$Q = I \tag{3}$$

which can be extended to more general cases without difficulty.

The systems considered in this paper are assumed to possess distinct complex eigenvalues. Both single-input and multi-input systems are considered. Numerical results are presented by means of examples in structural control of a multistory structure.

II. A Sequential Procedure of Modal Control Design

II.1. *Alteration of One Pair of Eigenvalues*

Consider first the case of altering one pair of conjugate complex eigenvalues for single-input systems. Equation (1) is now written in the form

$$\dot{x} = Ax + bu \tag{4}$$

where x is an n-vector, A is $n \times n$, b is an $n \times 1$ column vector and u is a scalar. It is assumed that the controllability condition is satisfied. Suppose that the conjugate complex eigenvalues of matrix A to be altered are

$$\lambda_{1,2} = \xi_1 \pm j\xi_2 \,. \tag{5}$$

Let C_j and D_j ($j = 1, 2, \ldots, n$) be corresponding eigenvectors of matrices A and A^T, respectively. It is well known [6] that

$$C_i^T D_j = \delta_{ij} \,, \qquad i, j = 1, \ldots, n. \tag{6}$$

Writing

$$C_{1,2} = C_1^1 \pm jC_1^2, \qquad D_{1,2} = D_1^1 \pm jD_1^2 \tag{7}$$

the inner products of C_1^i and D_1^i satisfy the relations

$$\begin{gathered} C_1^{1T}D_1^1 = \tfrac{1}{2}, \qquad C_1^{2T}D_1^2 = -\tfrac{1}{2} \\ C_1^{1T}D_1^2 = C_1^{2T}D_1^1 = 0. \end{gathered} \tag{8}$$

Let the linear feedback law be designed as

$$u = Kx = (\alpha_1 D_1^1 + \alpha_2 D_1^2)^T x. \tag{9}$$

We wish to determine α_1 and α_2 such that the new dynamic matrix $A_1 = A + bK$ has the desired eigenvalues, say,

$$\rho_{1,2} = \eta_1 \pm j\eta_2 \tag{10}$$

in place of λ_1 and λ_2.

As is shown in Appendix A, the orthogonality relations of vectors C_i and D_j ensure that λ_1 and λ_2 and their corresponding eigenvectors C_1 and C_2 are altered while all other eigenvalues and corresponding eigenvectors of A remain undisturbed.

If C_{11} and C_{12} are eigenvectors associated with ρ_1 and ρ_2, they can be expressed in terms of C_j by

$$C_{1k} = \sum_{j=1}^{n} g_{kj} C_j, \qquad k = 1, 2. \tag{11}$$

The vector b can be similarly expressed by

$$b = \sum_{j=1}^{n} p_j C_j \tag{12}$$

where

$$p_j = b^T D_j. \tag{13}$$

Using Eqs. (11) and (12), the relation $A_1 C_{11} = \rho_1 C_{11}$ and the linear independence of eigenvectors C_j yield

$$\lambda_j g_{1j} + \tfrac{1}{2}[g_{11}(\alpha_1 - j\alpha_2) + g_{12}(\alpha_1 + j\alpha_2)]\, p_j = \rho_1 g_{1j}\,. \tag{14}$$

The assumption that the system is controllable implies that $p_j \neq 0$ for all j. If we choose g_{1j} as

$$g_{1j} = p_j/(\rho_1 - \lambda_j) \tag{15}$$

for all j, then we have the relation

$$\tfrac{1}{2}[(\alpha_1 - j\alpha_2)\, p_1/(\rho_1 - \lambda_1) + (\alpha_1 + j\alpha_2)\, p_2/(\rho_1 - \lambda_2)] = 1. \tag{16}$$

Solving the above equation for α_1 and α_2 yields

$$\alpha_1 = \frac{e_{11}v_{12} + e_{12}v_{11}}{(v_{11}^2 + v_{12}^2)\, \xi_2}, \qquad \alpha_2 = \frac{e_{12}v_{12} - e_{11}v_{11}}{(v_{11}^2 + v_{12}^2)\, \xi_2}, \tag{17}$$

where

$$\begin{aligned} e_{11} &= (\eta_1 - \xi_1)^2 + \eta_2{}^2 - \xi_2{}^2, \\ e_{12} &= 2(\eta_1 - \xi_1)\, \xi_2, \\ v_{11} &= b^T D_1{}^1, \\ v_{12} &= b^T D_1{}^2. \end{aligned} \tag{18}$$

The uniqueness of the above design procedure is shown in Appendix A. Since ρ_1 and ρ_2 can be chosen arbitrarily, the design procedure permits an arbitrary change of the real or imaginary part or both of a pair of conjugate complex eigenvalues.

In summary, we have the following theorem.

THEOREM 1. *Suppose the time-invariant system defined by Eq.* (4) *is controllable. The gains of linear feedback law given by Eq.* (9) *to alter a pair of conjugate complex eigenvalues* $\lambda_{1,2}$ *to the desired eigenvalues* $\rho_{1,2}$ *are given by Eqs.* (17) *and* (18) *and they are unique.*

It is seen from Eqs. (9), (17) and (18) that the eigenvectors of A^T play a central role in the computation of the feedback matrix K.

II.2. *Alteration of More than One Pair of Eigenvalues*

Single-input systems. For altering more than one pair of conjugate complex eigenvalues by means of a single input, we consider a procedure which is developed by repeated applications of the technique advances in Section II.1. Essentially, it is equivalent to dividing the input into several parts, each part being responsible for one pair of conjugate complex eigenvalues.

Let (λ_1, λ_2) and (λ_3, λ_4) be two pairs of conjugate complex eigenvalues associated with the system matrix A to be altered, and let (ρ_1, ρ_2) and (ρ_3, ρ_4) be the desired replacement pairs. The alteration of the first pair has been carried out in Section II.1. For the second pair, we need to consider the dynamic system after the first control, i.e.,

$$\dot{x} = A_1 x + b u_1 \tag{19}$$

where $A_1 = A + bK$ with K given by Eq. (9). Following the procedure developed in Section II.1, we have

$$u_1 = (\alpha_3 D_{13}^1 + \alpha_4 D_{13}^2)^T x \tag{20}$$

where $D_{13} = D_{13}^1 + jD_{13}^2$ is the eigenvector corresponding to λ_3 of the matrix A_1.

In the above, the coefficients α_3 and α_4 take the forms

$$\begin{aligned} \alpha_3 &= \frac{e_{23}v_{24} + e_{24}v_{23}}{(v_{23}^2 + v_{24}^2)\,\xi_4}, \\ \alpha_4 &= \frac{e_{24}v_{24} - e_{23}v_{23}}{(v_{23}^2 + v_{24}^2)\,\xi_4}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} e_{23} &= (\eta_3 - \xi_3)^2 + \eta_4{}^2 - \xi_4{}^2, \\ e_{24} &= 2(\eta_3 - \xi_3)\,\xi_4, \\ v_{23} &= b^T D_{13}^1, \\ v_{24} &= b^T D_{13}^2. \end{aligned} \tag{22}$$

Summarizing, let the system be described by

$$\dot{x} = Ax + bu. \tag{23}$$

The single-input control for altering two pairs of eigenvalues (λ_1, λ_2) and (λ_3, λ_4) associated with the system matrix A to (ρ_1, ρ_2) and (ρ_3, ρ_4) is given by

$$u = (\alpha_1 D_1{}^1 + \alpha_2 D_1{}^2 + \alpha_3 D_{13}^1 + \alpha_4 D_{13}^2)^T x \tag{24}$$

where α_1 and α_2 are given by Eq. (17) and α_3 and α_4 are given by Eq. (21). Extensions to cases of altering more than two pairs of eigenvalues are obvious. As is shown in Appendix A, this design procedure is unique.

Computationally, it is seen that an efficient procedure is needed to determine eigenvectors associated with the transpose of the system matrices $A, A_1, A_2, \ldots$ after successive single-pair controls. This is developed in Appendix B, which gives a simple recursive procedure.

The results given above can be summarized in the following theorem:

THEOREM 2. *Suppose the time-invariant system defined by Eq.* (23) *is controllable. The gains of linear feedback law given by Eq.* (24) *to alter two pairs of conjugate complex eigenvalues* (λ_1, λ_2) *and* (λ_3, λ_4) *to the desired eigenvalues* (ρ_1, ρ_2) *and* (ρ_3, ρ_4) *are given by Eqs.* (17) *and* (21) *and they are unique.*

Multi-input systems. Consider a dynamic system described by

$$\dot{x} = Ax + Bu. \tag{25}$$

A multi-input control scheme can be generated for the single-input procedure given above under mild assumptions imposed on the control matrix B. It is noted that uniqueness of control design does not exist here and, in some cases, multi-input controls become degenerate when used to alter only one pair of eigenvalues [7].

Consider again the case of altering two pairs of eigenvalues descussed above. In this case, writing $B = [b_1\, b_2]$ and $u^T = [u_1\, u_2]$ and assuming that $b_1{}^T D_1 \neq 0$ and $b_2{}^T D_3 \neq 0$, where D_1 and D_3 are eigenvectors corresponding to λ_1 and λ_3 of matrix A^T, the controls u_1 and u_2 are then given by

$$\begin{aligned} u_1 &= (\alpha_1 D_1{}^1 + \alpha_2 D_1{}^2)^T x, \\ u_2 &= (\alpha_3 D_{13}^1 + \alpha_4 D_{13}^2)^T x, \end{aligned} \tag{26}$$

where α_1 and α_2 are defined by Eqs. (17) and (18) with b_1 replacing b and α_3 and α_4 defined by Eqs. (21) and (22) with b_2 replacing b.

As is shown in Appendix B, the eigenvector components D_{13}^1 and D_{13}^2 can be expressed in terms of those associated with the original system matrix A^T.

III. OPTIMAL CONTROLLER PLACEMENT

III.1. *Single-Input Systems*

In this section, a method of determining an optimal control matrix for single-input systems is developed. Following Section II.1, let $\lambda_{1,2} = \xi_1 \pm j\xi_2$ be the eigenvalues to be replaced by the desired values $\rho_{1,2} = \eta_1 \pm j\eta_2$. The control gain K is given by Eq. (9). In order to express $u(t)$ as a function of the control vector b explicitly, it can be written in the form

$$u(t) = \mathrm{Re}\{(\alpha_1 - j\alpha_2)\,[\beta_1(t)/(e_{11} + je_{12}) + \beta_2(t)/(e_{11} - je_{12})]\} \tag{27}$$

where Re( ) denotes the real part and $\beta_1(t)$ and $\beta_2(t)$ are functions of system dynamics and possible external excitations but not functions of b. In Eq. (27) only α_1 and α_2 are functions of b as given by Eqs. (17) and (18).

In the single-input case, it is assumed that $b^T b = 1$ without loss of generality. The substitution of Eq. (27) into Eq. (2) (with $Q = 1$) shows that E is a quadratic function in α_1 and α_2. In order to minimize the energy required, it is clear that a simple criterion can be established for choosing b by requiring that $\alpha_1^2 + \alpha_2^2$ is minimized subject to the controllability condition and to the constraint $b^T b = 1$.

It is seen from Eq. (17) that, since

$$\alpha_1^2 + \alpha_2^2 = (e_{11}^2 + e_{12}^2)/(v_{11}^2 + v_{12}^2)\, \xi_2^2 \tag{28}$$

an equivalent criterion is that $v_{11}^2 + v_{12}^2$ be maximized.

The result given above can now be stated below as a theorem. Let us first give the following definition.

Definition. The quantity $w_1 = v_{11}^2 + v_{12}^2$ as defined by Eq. (18) is called the energy coefficient for system defined by Eq. (4) for altering a pair of conjugate complex eigenvalues of the matrix A. It is noted that v_{11} and v_{12} are projections of the control vector b onto the real and imaginary components of D_1, the eigenvector associated with A^T corresponding to the eigenvalue pair undergoing alteration.

Theorem 3. *Under the control law given in Eq.* (9), *a sufficient condition for optimal design of control vector b which minimizes energy E is that the energy coefficient w_1 be maximized, subject to the constraint that $b^T b = 1$ and that the matrix pair (A, b) satisfy the controllability condition.*

The extension of the above result to the case in which k pairs of eigenvalues are altered can be carried out following the sequential procedure developed in Section II. Consider the case where two pairs of conjugate complex eigenvalues, $\lambda_{1,2}$ and $\lambda_{3,4}$, are to be altered to $\rho_{1,2}$ and $\rho_{3,4}$. The control is given by Eq. (24). As a function of b, the control u is again expressible as a quadratic function of α_1, α_2, α_3, and α_4, where α_1 and α_2 are given in Eq. (17) and α_3 and α_4 in Eq. (21). As is shown in Appendix B, the quantity D_{13} in Eq. (22) can be expressed as a linear function of eigenvectors associated with A^T, A being the original system matrix.

It is now easy to deduce control result when more than two pairs of eigenvalues are to be altered. Following the same procedure as one used previously, we arrive at the following general result for determining the optimal control vector b.

Definition. For altering k pairs of eigenvalues, the energy coefficient for the system given by Eq. (4) is defined by

$$w_k = v_{11}^2 + v_{12}^2 + v_{13}^2 + v_{14}^2 + \cdots + v_{1(2k-1)}^2 + v_{1(2k)}^2 \tag{29}$$

where

$$v_{1(2k-1)} = b^T D^1_{2k-1}\,, \qquad v_{1(2k)} = b^T D^2_{2k-1}\,. \tag{30}$$

In other words, $v_{1(2k-1)}$ and $v_{1(2k)}$ are projections of the control vector b onto the real and imaginary components of D_{2k-1}, the eigenvector associated with A^T corresponding to the eigenvalue λ_{2k-1} undergoing alteration.

THEOREM 4. *For modal control of k pairs of eigenvalues in single-input systems, a sufficient condition for optimal design of control vector b which minimizes energy E is that the energy coefficient w_k be maximized subject to the constraint that $b^Tb = 1$ and that the matrix pair (A, b) satisfy the controllability condition.*

III.2. *Multi-input Systems*

The optimal design of a control matrix for multi-input systems is now developed. Consider first the design of an $n \times 2$ control matrix B given by

$$B = [b_1\ b_2]. \tag{31}$$

For control purposes, controllability conditions must be satisfied in the design process. However, as is shown in Section II.2, the matrix B may be designed in such a way that each column b_j of the matrix B is determined by single-mode control. Assuming that two pairs of eigenvalues are to be altered, then b_1 may be designed to alter only the first pair and b_2 the second. Let

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{32}$$

The controls u_1 and u_2 are then given by Eqs. (26).

The procedure for obtaining the criterion for optimal design of a control matrix now follows that used for single-input systems. Generalizing to the case of altering k pairs of conjugate complex eigenvalues and again noting that D_{13} can be expressed as a linear function of eigenvectors associated with the transpose of matrix A, we have the following result.

THEOREM 5. *For modal control of k pairs of eigenvalues with $B = [b_1\, b_2 \ldots b_k]$, a sufficient condition for optimal design of control matrix B which minimizes energy E is that the energy coefficient w_k be maximized subject to the constraints that $b_j{}^Tb_j = 1$, $j = 1,\ldots, k$, and that the matrix pair (A, B) satisfy the controllability condition.*

In the above, the energy E is defined by

$$E = \int_0^{T_0} u^T u\, dt \tag{33}$$

with

$$u = [u_1\, u_2 \ldots u_k]^T \tag{34}$$

and w_k is defined by Eq. (29) with

$$v_{k(2k-1)} = b_{2k-1}^T D_{2k-1}^1, \qquad v_{k(2k)} = b_{2k-1}^T D_{2k-1}^2. \tag{35}$$

IV. Numerical Examples

In this section, numerical examples are presented for the case of structural control of a three-story structure. Let the state vector be written in the form

$$x = [x_1\, x_2\, x_3\, \dot{x}_1\, \dot{x}_2\, \dot{x}_3]^T \tag{36}$$

where x_j and $\dot{x}_j$ denote, respectively, displacement and velocity of the jth floor. An approximate equation of motion for the structure is [4]

$$\dot{x} = Ax + Bu + w \tag{37}$$

where u is the control vector and w is external excitation. To approximate a lightly damped structure, A is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.5 & 0.25 & 0 & -0.02 & 0.01 & 0 \\ 0.25 & -0.5 & 0.25 & 0.01 & -0.02 & 0.01 \\ 0 & 0.25 & -0.25 & 0 & 0.01 & -0.01 \end{bmatrix} \tag{38}$$

and

$$w = f(t)\,[0\ 0\ 0\ 1\ 1\ 1]^T. \tag{39}$$

The eigenvalues of the matrix A are

$$\begin{aligned} \lambda_{1,2} &= -0.001 \pm j0.2225, \\ \lambda_{3,4} &= -0.0078 \pm j0.6234, \\ \lambda_{5,6} &= -0.0162 \pm j0.9008. \end{aligned} \tag{40}$$

and the corresponding normalized eigenvectors are

$$\begin{aligned} C_{1,2} &= C_1^1 \pm jC_1^1, \\ C_{3,4} &= C_2^1 \pm jC_2^2, \\ C_{5,6} &= C_3^1 \pm jC_3^2. \end{aligned} \tag{41}$$

where

$$C_1^1 = \begin{bmatrix} 0.2420 \\ 0.4361 \\ 0.5438 \\ 0.0464 \\ 0.0836 \\ 0.1043 \end{bmatrix}, \qquad C_1^2 = \begin{bmatrix} -0.2096 \\ -0.3777 \\ -0.4710 \\ 0.0541 \\ 0.0974 \\ 0.1215 \end{bmatrix},$$

$$C_2^1 = \begin{bmatrix} 0.6218 \\ 0.2707 \\ -0.4986 \\ -0.0468 \\ -0.0208 \\ 0.0375 \end{bmatrix}, \qquad C_2^2 = \begin{bmatrix} 0.0673 \\ 0.2299 \\ -0.0539 \\ 0.3871 \\ 0.1723 \\ -0.3104 \end{bmatrix}, \tag{42}$$

$$C_3^1 = \begin{bmatrix} 0.4311 \\ -0.5376 \\ 0.2393 \\ -0.0820 \\ 0.1022 \\ -0.0455 \end{bmatrix}, \qquad C_3^2 = \begin{bmatrix} 0.0832 \\ -0.1038 \\ 0.0462 \\ 0.3870 \\ -0.4826 \\ 0.2148 \end{bmatrix}.$$

The quantities λ_j, $j = 1,\ldots, 6$, are also eigenvalues of A^T but the corresponding eigenvectors of A^T are

$$\begin{aligned} D_{1,2} &= D_1^1 \pm jD_1^2, \\ D_{3,4} &= D_2^1 \pm jD_2^2, \\ D_{5,6} &= D_3^1 \pm jD_3^2, \end{aligned} \tag{43}$$

where

$$D_1^1 = \begin{bmatrix} -0.0687 \\ -0\ 1238 \\ -0.1544 \\ -0.0858 \\ -0.1545 \\ -0.1907 \end{bmatrix}, \qquad D_1^2 = \begin{bmatrix} -0.0188 \\ -0.0338 \\ -0.0422 \\ 0.3085 \\ 0.5558 \\ 0.6931 \end{bmatrix},$$

$$D_2^1 = \begin{bmatrix} -0.3494 \\ -0.1555 \\ 0.2802 \\ 0.2707 \\ 0.1205 \\ -0.2171 \end{bmatrix}, \qquad D_2^2 = \begin{bmatrix} 0.1731 \\ 0.0771 \\ -0.1388 \\ 0.5638 \\ 0.2510 \\ -0.4521 \end{bmatrix}, \tag{44}$$

$$D_3^1 = \begin{bmatrix} 0.3841 \\ -0.4790 \\ 0.2132 \\ 0.1128 \\ -0.1407 \\ 0.0626 \end{bmatrix}, \quad D_3^2 = \begin{bmatrix} 0.0947 \\ -0.1181 \\ 0.0526 \\ -0.4244 \\ 0.5292 \\ -0.2359 \end{bmatrix}.$$

Let us assume that

$$f(t) = 3 \sin 0.25t + 5 \sin 0.35t + 7 \sin 1.25t + 4 \sin 1.5t \tag{45}$$

whose first frequency is close to that of the system. The control objective is to reduce excessive displacements by applying control so that the system matrix of the resulting system ha eigenvalues $\rho_{1,2}$, $\rho_{3,4}$, and $\lambda_{5,6}$ where

$$\begin{aligned} \rho_{1,2} &= -0.099 \pm j4, \\ \rho_{3,4} &= -0.0778 \pm j0.6234. \end{aligned} \tag{46}$$

EXAMPLE 1. Consider the case of placing two controllers on two of the three floors. In the single-input case, Theorem 4 immediately determines the best locations to be the second and third floors with

$$b_{2,3} = [0 \ 0 \ 0 \ 0 \ 0.4766 \ 0.8791]^T. \tag{47}$$

TABLE I

Energy Coefficients and Energy in Example 1

Control vector b	Energy coefficient w_2	Energy E ($T_0 = 50$)
$b_{1,3}$	0.7923	55,968
$b_{1,2}$	0.8476	42,335
$b_{2,3}$	0.9184	33,382

The values of its associated energy coefficient w_2 and energy E are compared with the two other possible control configurations, $b_{1,2}$ and $b_{1,3}$, and are given in Table I. Figure 1 also gives the displacements of the top floor under all control schemes. It is seen that the optimal control design also gives optimal performance in terms of displacement reduction. Similar results are also obtained for floor acceleration reductions. Hence, from the standpoint of safety and

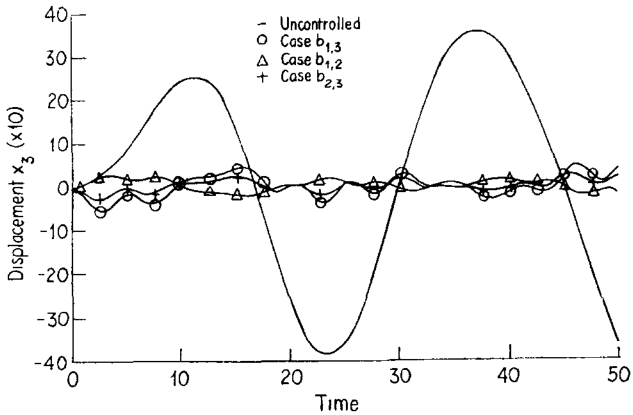

FIG. 1. Displacement X_3 in Example 1.

comfort control [3] as well as energy consideration, the choice $b_{2,3}$ is uniformly superior in the single-input case.

EXAMPLE 2. The same problem is now considered using a two-loop control scheme. Let us determine b_1 and b_2 of the matrix B such that b_1 is designed to alter only the first pair of complex conjugate eigenvalues, $\lambda_{1,2}$, and b_2 is designed to alter the second pair, $\lambda_{3,4}$.

Under these assumptions, the following designs of control matrix are of practical interest.

Case A. Different control forces applied to first and third floors:

$$B_{1,3} = \begin{bmatrix} 0 & 0 & 0 & 0.6256 & 0 & 0.7801 \\ 0 & 0 & 0 & 0.9136 & 0 & -0.4060 \end{bmatrix}^T.$$

Case B. Different control forces applied to first and second floors:

$$B_{1,2} = \begin{bmatrix} 0 & 0 & 0 & -0.4066 & 0.9130 & 0 \\ 0 & 0 & 0 & 0.8744 & -0.4852 & 0 \end{bmatrix}^T.$$

Case C. Different control forces applied to second and third floors:

$$B_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.8744 & 0.4852 \\ 0 & 0 & 0 & 0 & 0.7801 & -0.6256 \end{bmatrix}^T.$$

All three designs satisfy the controllability conditions.

As predicted by Theorem 5 and verified by Table II, Case C produces the optimal control configuration and, as shown in Fig. 2, it also is the most effective in reduction of displacement. Calculations also show that Case C is best in reduction of acceleration as well.

TABLE II

Energy Coefficients and Energy in Example 2

Control matrix B	Energy coefficient w_2	Energy E ($T_0 = 50$)
Case A	1.1809	24,615
Case B	0.3271	90,616
Case C	1.2946	19,627

Furthermore, a comparison of values in Tables I and II shows that Case C in two-loop control scheme is best overall under energy criterion. The energy coefficient in this case is maximum and it implies that energy coefficient can be used as an effectiveness measure crossing single-input and multi-input lines.

V. Concluding Remarks

Within the framework of modal control, a simple criterion for optimal design of the control matrix has been established under energy constraint. In control of large systems, the question of where to place a limited number of controllers is of practical importance and this result provides a simple guide for making this determination when energy required is to be minimized. Although only complex eigenvalues are considered, it is easily applied to cases involving real eigenvalues.

It has also been shown through examples that minimum-energy control also

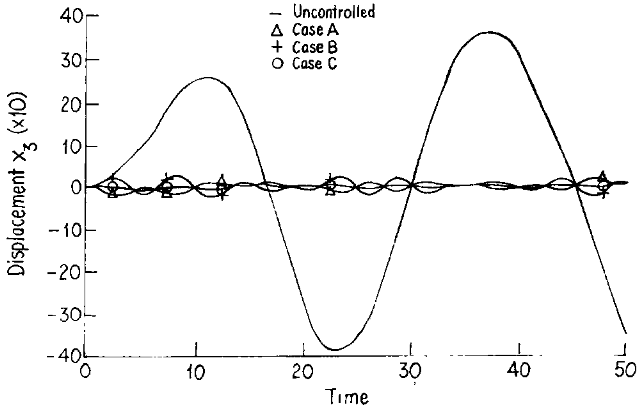

Fig. 2. Displacement X_3 in Example 2.

leads to most effective control in these cases, a desirable by-product of the analysis. Also it is seen that the energy coefficients for both single-input and multi-input cases take the same form and thus can be used to determine overall optimal control configuration.

It is worth pointing out that, under the energy criterion, the analysis presented herein provides some guide in determining the desired locations of the altered eigenvalues. Consider, for example, the case where the eigenvalue λ_1 is to be altered to ρ_1 using the single-input scheme. In general, some freedom exists in the choice of ρ_1 beyond requirements such as $|\rho_1 - \lambda_1| \geqslant \epsilon$.

Equation (28) gives

$$\alpha_1{}^2 + \alpha_2{}^2 = (e_{11}^2 + e_{12}^2)/(v_{11}^2 + v_{12}^2)\, \xi_2{}^2. \tag{48}$$

After control vector b has been determined from Theorem 3, the denominator of Eq. (48) is fixed and the numerator can be expressed by

$$e_{11}^2 + e_{12}^2 = (\eta_1 - \xi_1)^4 + (\eta_1 - \xi_1)^2\,[4 - 2(n_2{}^2 - \xi_2{}^2)] + (\eta_2{}^2 - \xi_2{}^2)^2. \tag{49}$$

Since $\alpha_1{}^2 + \alpha_2{}^2$ is to be minimized for minimum energy, the most energy-efficient locations of η_1 and η_2, the real and imaginary parts of ρ_1, can be determined by minimizing Eq. (49) with respect to η_1 and η_2. This is to be carried out, of course, under constraints such as $|\rho_1 - \lambda_1| \geqslant \epsilon$.

Appendix A

The object of this appendix is to prove that the design procedures presented in Sections II.1 and II.2 are unique for single-input systems. It is also shown that the eigenvectors corresponding to those unaltered eigenvalues of matrix A remain unaltered.

Suppose that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of the system matrix A are to be replaced by desired value $\rho_1, \rho_2, \ldots, \rho_r$. Let

$$u = Kx \tag{A1}$$

and let K_i be the ith row of the matrix K. Expressing $K_i{}^T$ in terms of eigenvectors of matrix A^T, we have

$$K_i^T = \sum_{j=1}^{n} \alpha_{ij} D_j\,. \tag{A2}$$

Denoting the eigenvalues of $A_1 = A + BK$ by ρ_{1i}, $i = 1, \ldots, n$, we have

$$\begin{aligned} \rho_{1i} &= \rho_i\,, \qquad i = 1, \ldots, r, \\ &= \lambda_i\,, \qquad \text{otherwise.} \end{aligned} \tag{A3}$$

The corresponding eigenvectors C_{1i} of ρ_{1i} can be expressed as [see Eq. (11)]

$$C_{1i} = \sum_{j=1}^{n} g_{ij} C_j \tag{A4}$$

and, following Eq. (14), we can write

$$\lambda_j g_{ij} + \sum_{k=1}^{n} \left(\sum_{m=1}^{n} \alpha_{km} g_{im} \right) p_{kj} = \rho_{1i} g_{ij} \tag{A5}$$

where $p_{kj} = b_k^T D_j$, b_k being the kth column of matrix B. Since $\rho_{1i} = \lambda_i$ for $i > r$, this gives

$$\sum_{k=1}^{r} \left(\sum_{m=1}^{n} \alpha_{km} g_{im} \right) p_{kj} = 0, \qquad i > r \text{ and all } j. \tag{A6}$$

Consider an n-dimensional vector p_k whose components are p_{kj}. Let us assume that matrix B has full rank without loss of generality. The controllability condition implies $p_k \neq 0$ for all k. Then,

$$\sum_{m=1}^{n} \alpha_{km} g_{im} = 0. \tag{A7}$$

The substitution of Eq. (A7) into Eq. (A5) gives

$$\lambda_j g_{ij} = \lambda_i g_{ij}, \qquad i > r \quad \text{and all } j. \tag{A8}$$

The assumption that the system possesses distinct eigenvalues implies that

$$g_{ij} = 0, \qquad i > r, \quad i \neq j, \tag{A9}$$

which leads to

$$C_{1i} = g_{ii} C_i, \qquad i > r, \tag{A10}$$

which shows that the eigenvectors corresponding to unaltered eigenvalue of the system matrix A remain unaltered for single-input as well as multi-input systems.

The above result shows that

$$BKC_j = 0, \qquad j > r, \tag{A11}$$

and

$$K_i C_j = 0, \qquad j > r, \tag{A12}$$

since B has full rank. Thus, the row vector K_i can be expressed as a linear combination of eigenvectors associated with those altered eigenvalues, i.e.,

$$K_i^T = \sum_{j=1}^{r} \alpha_{ij} D_j \tag{A13}$$

for all i. For single-input systems, $i = 1$ and we can write

$$K^T = \sum_{j=1}^{r} \alpha_j D_j \tag{A14}$$

where α_j can be determined by solving the system of linear algebraic equations

$$\sum_{j=1}^{r} \alpha_j p_j/(\rho_i - \lambda_j) = 1, \qquad i = 1, 2,..., r. \tag{A15}$$

It is clear that α_j as determined from Eq. (A15) is unique. Hence, K is unique for single-input systems.

APPENDIX B

Referring to Section II.2, denoting A_r as the system matrix after altering r pairs of eigenvalues sequentially, it is of interest to determine the eigenstructure of A_r^T in terms of that associated with the original system matrix A.

Consider first the determination of eigenvectors associated with A_1^T. Results in Section II.1 show that the eigenvectors of A_1 are

$$\begin{aligned} C_{1j} &= \sum_{k=1}^{n} p_k C_k/(\rho_{1j} - \lambda_k), && j = 1, 2, \\ &= C_j, && \text{otherwise}. \end{aligned} \tag{B1}$$

with corresponding eigenvalues

$$\begin{aligned} \rho_{1j} &= \rho_j, && j = 1, 2, \\ &= \lambda_j, && \text{otherwise}. \end{aligned} \tag{B2}$$

As seen in Section II.1, the eigenvectors D_{1j} of A_1^T may be selected such that

$$D_{1j}^T C_{1i} = \delta_{ij}, \qquad i, j = 1, 2,..., n. \tag{B3}$$

Clearly, D_{1j} can be written in the form

$$\begin{aligned} D_{1j} &= a_{1j} D_1 + b_{1j} D_2, && j = 1, 2, \\ &= a_{1j} D_1 + b_{1j} D_2 + D_j, && \text{otherwise}. \end{aligned} \tag{B4}$$

Let us first calculate D_{1j} for $j \geqslant 3$. The substitution of Eq. (B4) into (B3) gives two simultaneous equations for each $j \geqslant 3$ from which the coefficients a_{1j} and b_{1j} can be determined. The result is

$$\begin{aligned} a_{1j} &= p_j s_{1j}/p_1, \qquad j \geqslant 3, \\ b_{1j} &= p_j t_{1j}/p_2, \qquad j \geqslant 3, \end{aligned} \tag{B5}$$

where

$$\begin{aligned} s_{1j} &= q[1/(\rho_{11} - \lambda_2)(\rho_{12} - \lambda_j) - 1/(\rho_{11} - \lambda_j)(\lambda_{12} - \lambda_2)], \\ t_{1j} &= q[1/(\rho_{11} - \lambda_j)(\rho_{12} - \lambda_1) - 1/(\rho_{12} - \lambda_j)(\rho_{11} - \lambda_1)], \\ q &= 1/[1/(\rho_{11} - \lambda_1)(\rho_{12} - \lambda_2) - 1/(\rho_{12} - \lambda_1)(\rho_{11} - \lambda_2)]. \end{aligned} \tag{B6}$$

Similarly, Eq. (B3) for the case of $j = 1, 2$ leads to

$$\begin{aligned} a_{1j} &= s_{1j}/p_1, \qquad j = 1, 2, \\ b_{1j} &= t_{1j}/p_2, \qquad j = 1, 2, \end{aligned} \tag{B7}$$

where

$$\begin{aligned} s_{1j} &= q[\delta_{1j}/(\rho_{12} - \lambda_2) - \delta_{2j}/(\rho_{11} - \lambda_2)], \\ t_{1j} &= q[\delta_{2j}/(\rho_{11} - \lambda_1) - \delta_{1j}/(\rho_{12} - \lambda_1)]. \end{aligned} \tag{B8}$$

In summary, the eigenvectors D_{1j} of A_1^T have the forms

$$\begin{aligned} D_{1j} &= s_{1j} D_1/p_1 + t_{1j} D_2/p_2, && j = 1, 2, \\ &= p_j s_{1j} D_1/p_1 + p_j t_{1j} D_2/p_2 + D_j, && \text{otherwise}, \end{aligned} \tag{B9}$$

where s_{1j} and t_{1j} are given by Eqs. (B6) and (B8).

For the general case, we can give a matrix representation of the eigenvectors of matrix A_r^T. The matrix representation of Eq. (B9) is

$$\phi_1 = \phi_0 \Omega_1 \tag{B10}$$

where

$$\begin{aligned} \phi_1 &= [D_{11}\ D_{12}\ \cdots\ D_{1n}], \\ \phi_0 &= [D_1\ D_2\ \cdots\ D_n], \\ \Omega_1 &= \begin{bmatrix} v_1 & w_1 \\ 0_{n-2} & I_{n-2} \end{bmatrix}, \end{aligned} \tag{B11}$$

with

$$\begin{aligned} v_1 &= \begin{bmatrix} s_{11}/p_1 & s_{12}/p_2 \\ t_{11}/p_1 & t_{12}/p_2 \end{bmatrix}, \\ w_1 &= \begin{bmatrix} s_{13}p_3/p_1 & \cdots & s_{1n}p_n/p_1 \\ t_{13}p_3/p_1 & \cdots & t_{1n}p_n/p_1 \end{bmatrix}. \end{aligned} \tag{B12}$$

In general, after altering r pairs of eigenvalues of A, the eigenvector matrix ϕ_r of A_r^T can be written as

$$\phi_r = \phi_0 \prod_{k=1}^{r} \Omega_k \tag{B13}$$

where

$$\Omega_k = \begin{bmatrix} I_{2k-2} & 0 & 0 \\ u_k & v_k & w_k \\ 0 & 0 & I_{n-2k} \end{bmatrix} \tag{B14}$$

$$u_k = \begin{bmatrix} \dfrac{s_{k1}p_{k1}}{p_{k(2k-1)}} & \dfrac{s_{k2}p_{k2}}{p_{k(2k)}} & \cdots & \dfrac{s_{k(2k-2)}p_{k(2k-2)}}{p_{k(2k)}} \\ \dfrac{t_{k1}p_{k1}}{p_{k(2k-1)}} & \dfrac{t_{k2}p_{k2}}{p_{k(2k)}} & \cdots & \dfrac{t_{k(2k-2)}p_{k(2k-2)}}{p_{k(2k)}} \end{bmatrix}, \tag{B15}$$

$$v_k = \begin{bmatrix} \dfrac{s_{k(2k-1)}}{p_{k(2k-1)}} & \dfrac{s_{k(2k)}}{p_{k(2k)}} \\ \dfrac{t_{k(2k-1)}}{p_{k(2k-1)}} & \dfrac{t_{k(2k)}}{p_{k(2k)}} \end{bmatrix}, \tag{B16}$$

$$w_k = \begin{bmatrix} \dfrac{s_{k(2k+1)}p_{k(2k+1)}}{p_{k(2k-1)}} & \cdots & \dfrac{s_{kn}p_{kn}}{p_{k(2k-1)}} \\ \dfrac{t_{k(2k+1)}p_{k(2k+1)}}{p_{k(2k-1)}} & \cdots & \dfrac{t_{kn}p_{kn}}{p_{k(2k-1)}} \end{bmatrix}, \tag{B17}$$

$$p_{kj} = b^T D_{(k-1)j}\,. \tag{B18}$$

In the above, $D_{(k-1)j}$ are the eigenvectors of A_{k-1} and

$$\begin{aligned} s_{kj} &= q_k[\delta_{2k(j)}/(\rho_{k(2k-1)} - \lambda_{2k}) - \delta_{(2k-1)j}/(\rho_{k(2k)} - \lambda_{2k})], \qquad j = 2k-1, 2k, \\ &= q_k[1/(\rho_{k(2k)} - \lambda_j)\,(\rho_{k(2k-1)} - \lambda_{2k}) - 1/(\rho_{k(2k-1)} - \lambda_j)\,(\rho_{k(2k)} - \lambda_{2k})], \end{aligned} \tag{B19}$$

otherwise,

$$\begin{aligned} t_{kj} &= q_k[\delta_{(2k-1)j}/(\rho_{k(2k)} - \lambda_{2k-1}) - \delta_{2k(j)}/(\rho_{k(2k-1)} - \lambda_{2k-1})], \qquad j = 2k-1, 2k, \\ &= q_k[1/(\rho_{k(2k-1)} - \lambda_j)\,(\rho_{k(2k)} - \lambda_{2k-1}) - 1/(\rho_{k(2k)} - \lambda_j)\,(\rho_{k(2k-1)} - \lambda_{(2k-1)})], \end{aligned} \tag{B20}$$

otherwise,

$$q_k = 1/[1/\rho_{k(2k-1)} - \lambda_{2k-1})\,(\rho_{k(2k)} - \lambda_{2k}) - 1/(\rho_{k(2k)} - \lambda_{2k-1})\,(\rho_{k(2k-1)} - \lambda_{2k})]. \tag{B21}$$

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